New Generating Relations for Two Dimensional Hermite Polynomials

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Abstract: In this paper, we derive generating relations involving two dimensional Hermite polynomials $H_{nm}^{(R)}(y_1, y_2)$ by using Lie-algebraic method. Some new and known generating relations related to $H_{nm}^{(R)}(y_1, y_2)$ are also obtained.

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1. Introduction

The Hermite polynomials of several variables arise quite naturally in almost all problems relating to quantum systems described by means of multidimensional quadratic Hamiltonians [3-6]. The two dimensional Hermite polynomials $H_{nm}^{(R)}(y_1, y_2)$ are defined by means of the generating function [2]

$$\exp\left[-\frac{1}{2}aRa + aRy\right] = \sum_{n,m=0}^{\infty} \frac{a_1^n a_2^m}{n!\, m!} H_{nm}^{(R)}(y). \quad (1.1)$$

Here a_1 and a_2 are arbitrary complex numbers combined into vector $a = (a_1, a_2)$.

$$aRa = \sum_{i,k=0}^{2} , a_i R_{ik} a_k$$

$$aRy = \sum_{i,k=0}^{2} , a_i R_{ik} y_k$$

and R is the symmetric matrix

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{pmatrix}$$

The two dimensional Hermite polynomials $H_{nm}^{(R)}(y_1, y_2)$ defined by Eq.(1.1) satisfy the following differential and pure recurrence relations:

$$\frac{\partial}{\partial y_1} H_{nm}^{(R)}(y_1, y_2) = R_{11} n H_{n-1m}^{(R)}(y_1, y_2) + R_{12} m H_{nm-1}^{(R)}(y_1, y_2),$$

$$\frac{\partial}{\partial y_2} H_{nm}^{(R)}(y_1, y_2) = R_{12} n H_{n-1m}^{(R)}(y_1, y_2) + R_{22} m H_{nm-1}^{(R)}(y_1, y_2),$$

$$H_{n+1m}^{(R)}(y_1, y_2) = (R_{11} y_1 + R_{12} y_2) H_{nm}^{(R)}(y_1, y_2) - R_{11} n H_{n-1m}(y_1, y_2) - R_{12} m H_{nm-1}(y_1, y_2),$$

$$H_{nm+1}^{(R)}(y_1, y_2) = (R_{12} y_1 + R_{22} y_2) H_{nm}^{(R)}(y_1, y_2) - R_{12} n H_{n-1m}(y_1, y_2) - R_{22} m H_{nm-1}(y_1, y_2). \quad (1.2)$$

The differential equation for two dimensional Hermite polynomials $H_{nm}^{(R)}(y_1, y_2)$ is given as

$$\left(\frac{1}{R_{11} R_{22} - R_{12}^2}\left(-R_{22}\frac{\partial^2}{\partial y_1^2} + 2R_{12}\frac{\partial^2}{\partial y_1 \partial y_2} - R_{11}\frac{\partial^2}{\partial y_2^2}\right) + \left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right) - (n+m)\right) H_{nm}^{(R)}(y_1, y_2) = 0. \quad (1.3)$$

The Rodrigues type formula for two dimensional Hermite polynomials $H_{nm}^{(R)}(y_1, y_2)$ is given as

$$H_{nm}^{(R)}(y_1, y_2) = (-1)^{n+m} e^{(\frac{1}{2})yRy} \frac{\partial^{n+m}}{\partial y_1^n \partial y_2^m} e^{(-\frac{1}{2})yRy}. \quad (1.4)$$

In the quantum mechanical application for the case of the absence of an extranal force, $y = 0$. Suppose that $R_{11} \neq 0$ and $R_{22} \neq 0$, then eq.(1.1) gives

$$E = \exp\left[-\frac{1}{2}R_{11} a_1^2 - \frac{1}{2}R_{22} a_2^2 - R_{12} a_1 a_2\right]. \quad (1.5)$$

After expanded in Taylor's series and introducing the notation

$$r = \frac{R_{12}}{\sqrt{R_{11} R_{22}}}, \alpha = \sqrt{\frac{R_{22} a_2}{R_{11} a_1}}$$

and using the associated Legender functions [2]

$$P_q^s(z) = \frac{(-1)^q}{2^q q!} (z^2 - 1)^{\frac{s}{2}} \frac{d^{q+s}}{dz^{q+s}} (1 - z^2)^q,$$

the formula for two dimensional Hermite polynomials of zero arguments is obtained as

$$H_{nm}^{(R)}(0,0) = \mu_{mn}! \,(-1)^{\frac{m+n}{2}} R_{11}^{\frac{n}{2}} R_{22}^{\frac{m}{2}} (r^2 - 1)^{\frac{m+n}{4}} P_{(m+n)/2}^{(m-n)/2} \left(\frac{r}{\sqrt{r^2 - 1}}\right). (1.6)$$

where μ_{mn} = min (m,n) and integers m,n must have the same parity otherwise the right hand side equals zero.

For coinciding indices

$$H_{nn}^{(R)}(0,0) = n! \,(-detR)^{\frac{n}{2}} P_n\left(\frac{-R_{12}}{\sqrt{-detR}}\right). (1.7)$$

where $P_n(z)$ being the Legender polynomial.

For non zero vector y function $H_{nm}^{(R)}(y_1, y_2)$ can be written as a finite sum of products of the usual Hermite polynomials [1]

$$\left(\frac{R_{11}^{\,n} R_{22}^{\,m}}{2^{m+n}}\right)^{-1/2} H_{nm}^{(R)}(y_1, y_2)$$

$$= \sum_{k=0}^{\mu_{mn}} \left(-\frac{2R_{12}}{\sqrt{R_{11}R_{22}}}\right)^k \frac{n!\,m!}{(n-k)!\,(m-k)!\,k!} H_{n-k}\left(\frac{f_1}{\sqrt{2R_{11}}}\right) H_{m-k}\left(\frac{f_2}{\sqrt{2R_{22}}}\right). (1.8)$$

where

$$f_1 = R_{11}y_1 + R_{12}y_2, \quad f_2 = R_{12}y_1 + R_{22}y_2. (1.9)$$

Immediately from the generating function (1.1), the expression for $H_{nm}^{(R)}(y_1, y_2)$ in terms of $H_{nm}^{(R)}(0,0)$ and variables f_1, f_2 defined by Eq.(1.9)is obtained as

$$H_{nm}^{(R)}(y_1, y_2) = \sum_{l=0}^{n} \sum_{k=0}^{m} {}_n{}_l \,{}_m{}_k \, H_{lk}^{(R)}(0,0) f_1^{\,n-l} f_2^{\,m-k}. (1.10)$$

`Consequently,$H_{nm}^{(R)}(y_1, y_2)$ is a polynomial of f_2 and f_2 with the coefficients expressed through the Gegenbauer polynomials.

In this paper, we consider the problem of framing $H_{nm}^{(R)}(y_1, y_2)$ into the context of the representation $\uparrow_{\omega,\mu}$ of four dimensional Lie algebra $G(0,1)$. In section 2 we obtain generating relations involving $H_{nm}^{(R)}(y_1, y_2)$ and associated Laguerre polynomials $L_n^\alpha(x)$. In Section 3, we consider some special cases which would yields many new and known generating relations.

2. Representation $\uparrow_{\omega,\mu}$ of $G(0,1)$ and generating relations

Within the group-theoretic context, indeed a given class of special functions appears as a set of matrix elements of irreducible representation of a given Lie group. The algebraic properties of the group are then reflected in the functional and differential equations satisfied by a given family of special functions, whilst the geometry of the homogeneous space determines the nature of the integral representation associated with the family.

We have the isomorphism $G(0,1) = L[G(0,1)]$ ([7]; p.36), where 4-dimensional complex local Lie group $G(0,1)$ is the set of all 4×4 matrices of the form ([7]; p.9)

$$g = \begin{pmatrix} 1 & ce^\tau & a & \tau \\ 0 & e^\tau & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, b, c, \tau \in \mathbb{C}, (2.1)$$

A basis for the Lie algebras $G(0,1) = L[G(0,1)]$ is provided by the matrices

$$J^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (2.2)$$

with commutations relations

$$[J^3, J^\pm] = \pm J^3, [J^+, J^-] = -E, [E, J^\pm] = [E, J^3] = 0, (2.3)$$

The irreducible representation $\uparrow_{\omega,\mu}$ of $G(0,1)$ is defined for each $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum S of this representation is the set $\{-\omega + k : k \text{ a nonnegative integer}\}$ and there is a basis $(f_m)_{m \in S}$ for the representation space V, In particular, we are looking for the functions $f_{nm}(y_1, y_2, q, s) = Z_{nm}^{(R)}(y_1, y_2) q^n s^m$ such that

$$J^3 f_{nm} = n f_{nm}, E f_n m = \mu f_{nm}, J^+ f_{nm} = \mu f_{n+1m}, J^- f_{nm} = (n + \omega) f_{n-1m},$$

$$C_{0,1} f_{nm} = (J^+ J^- - E J^3) f_{nm} = \mu \omega f_{nm}, \mu \neq 0. \quad (2.4)$$

for all $n \in S$

The commutation relations satisfied by the operators $J^3, J^\pm, E$ are

$$[J^3, J^\pm] = \pm J^\pm, [J^+, J^-] = -E, [E, J^\pm] = [E, J^3] = 0. (2.5)$$

The linear differential operators $J^{\pm}, J^3, E$ takes the form

$$J^+ = (R_{12}y_1 + R_{12}y_2)q - q\frac{\partial}{\partial y_1},$$

$$J^- = \frac{-1}{(R_{11}R_{22} - R_{12}{}^2)q}(R_{22}\frac{\partial}{\partial y_1} - R_{12}\frac{\partial}{\partial y_2}), \quad (2.6)$$

$$J^3 = q\frac{\partial}{\partial q},$$

$$E = 1,$$

and satisfy the commutation relations (2.5).

There is no loss of generality for special function theory if we set $w = 0$, $\mu = 1$, then in terms of the function $Z_{nm}^{(R)}(y_1, y_2)$, relation (2.4) reduce to

$$((R_{11}y_1 + R_{12}y_2) - \frac{\partial}{\partial y_1})Z_{nm}^{(R)}(y_1, y_2) = Z_{n+1m}^{(R)}(y_1, y_2),$$

$$\frac{1}{R_{11}R_{22} - R_{12}{}^2}(R_{22}\frac{\partial}{\partial y_1} - R_{12}\frac{\partial}{\partial y_2})Z_{nm}^{(R)}(y_1, y_2) = nZ_{n-1m}^{(R)}(y_1, y_2),$$

$$(\frac{\partial^2}{\partial y_1{}^2} - \frac{R_{12}}{R_{22}}\frac{\partial^2}{\partial y_1 \partial y_2} - (R_{11}y_1 + R_{12}y_2)(\frac{\partial}{\partial y_1} - \frac{R_{12}}{R_{22}}\frac{\partial}{\partial y_2})$$
$$+ \frac{n(R_{11}R_{22} - R_{12}{}^2)}{R_{22}})Z_{nm}^{(R)}(y_1, y_2) = 0, \quad (2.7)$$

Again if we take the functions $f_{nm}(y_1, y_2, q, s) = Z_{nm}^{(R)}(y_1, y_2)q^n s^m$ such that

$$J'^3 f_{nm} = mf_{nm}, E'f_nm = \mu f_{nm}, J'^+ f_{nm} = \mu f_{nm+1}, J'^- f_{nm}$$
$$= (m + \omega)f_{nm-1},$$

$$C'_{0,1}f_{nm} = (J'^+ J'^- - E'J'^3)f_{nm} = \mu\omega f_{nm}, \mu \neq 0. \quad (2.8)$$

for all $m \in S$, where the differential operators $J'^{\pm}, J'^3, E'$ are given by

$$J'^+ = (R_{12}y_1 + R_{22}y_2)s - s\frac{\partial}{\partial y_2},$$

$$J'^- = \frac{-1}{(R_{11}R_{22} - R_{12}{}^2)s}(R_{12}\frac{\partial}{\partial y_1} - R_{11}\frac{\partial}{\partial y_2}), \quad (2.9)$$

$$J'^3 = s\frac{\partial}{\partial s},$$

$$E' = 1,$$

and satisfy the commutation relations identical to (2.5).

just as before taking $w = 0$ and $\mu = 1$ relations (2.8) becomes

$$((R_{12}y_1 + R_{22}y_2) - \frac{\partial}{\partial y_2})Z_{nm}^{(R)}(y_1, y_2) = Z_{nm+1}^{(R)}(y_1, y_2),$$

$$\frac{-1}{R_{11}R_{22} - R_{12}{}^2}(R_{12}\frac{\partial}{\partial y_1} - R_{11}\frac{\partial}{\partial y_2})Z_{nm}^{(R)}(y_1, y_2)$$
$$= mZ_{nm-1}^{(R)}(y_1, y_2),$$

$$(\frac{\partial^2}{\partial y_2{}^2} - \frac{R_{12}}{R_{11}}\frac{\partial^2}{\partial y_2 \partial y_1} - (R_{12}y_1 + R_{22}y_2)(\frac{\partial}{\partial y_2} - \frac{R_{12}}{R_{11}}\frac{\partial}{\partial y_1})$$
$$+ \frac{m(R_{11}R_{22} - R_{12}{}^2)}{R_{11}})Z_{nm}^{(R)}(y_1, y_2) = 0. \quad (2.10)$$

We observe that from (2.7) and (2.10) that $Z_{nm}(y_1, y_2) = H_{nm}^{(R)}(y_1, y_2)$, where $H_{nm}^{(R)}(y_1, y_2)$ is given by (1.1) It follows that the functions $f_{nm}(y_1, y_2, q, s) = H_{nm}^{(R)}(y_1, y_2)q^n s^m$, $n \in S$ form a basis for a realization of the representation $\uparrow_{0,1}$ of $G(0,1)$. This representation of $G(0,1)$ can be extended to a local multiplier representation $T(g)$, $g \in G(0,1)$ defined on F, the space of all functions analytic in a neighbourhood of the point $(y_1{}^0, y_2{}^0, q^0, s^0) = (1,1,1,1)$.

Using operators (2.6), the local multiplier representation ([7];p.17) takes the form

$$[T(\exp a_1 E)f](y_1, y_2, q, s) = \exp(a_1)f(y_1, y_2, q, s),$$

$$[T(\exp b_1 J^+)f](y_1, y_2, q, s)$$
$$= \exp\left(\frac{qb_1}{2}(2R_{11}y_1 + 2R_{12}y_2 - b_1 R_{11}q)\right) f\left(y_1\left(1 - \frac{qb_1}{y_1}\right), y_2, q, s\right),$$

$$[T(\exp c_1 J^-)f](y_1, y_2, q, s)$$
$$= f\left(y_1\left(1 + \frac{c_1 R_{22}}{(R_{11}R_{22} - R_{12}{}^2)y_1 q}\right), y_2\left(1 - \frac{c_1 R_{12}}{(R_{11}R_{22} - R_{12}{}^2)y_2 q}\right), q, s\right),$$

$$[T(\exp \tau_1 J^3)f](y_1, y_2, q, s) = f(y_1, y_2, qe^{\tau_1}, s), \quad (2.11)$$

for $f \in F$. If $g \in G(0,1)$ has parameters (a_1, b_1, c_1, τ_1) then

$$T(g) = T(\exp a_1 E) T(\exp b_1 J^+) T(\exp c_1 J^-) T(\exp \tau_1 J^3)$$

and therefore we obtain

$$[T(g)f](y_1, y_2, q, s)$$
$$= \exp\left(a_1 + \frac{qb_1}{2}(2R_{11}y_1 + 2R_{12}y_2 - b_1 R_{11}q)\right)$$

$$\cdot f\left(y_1\left(1 - \frac{qb_1}{y_1} + \frac{c_1 R_{22}}{(R_{11}R_{22} - R_{12}{}^2)y_1 q}\right), y_2\left(1 - \frac{c_1 R_{12}}{(R_{11}R_{22} - R_{12}{}^2)y_2 q}\right), qe^{\tau_1}, s\right). \quad (2.12),$$

The matrix elements of $T(g)$ with respect to the analytic basis $f_{nm}(y_1,y_2,q,s) = H_{nm}^{(R)}(y_1,y_2)q^n s^m$, are the functions $A_{lk}(g)$, uniquely determined by $\uparrow_{0,1}$ of $G(0,1)$ and we obtain relations

$$[T(g)f_{k,m}](y_1,y_2,q,s) = \sum_{l=0}^{\infty} A_{lk}(g)f_{lm}(y_1,y_2,q,s), k = 0,1,2,\ldots, (2.13)$$

which on simplification yields

$$\exp\left(a_1 + \frac{qb_1}{2}(2R_{11}y_1 + 2R_{12}y_2 - b_1R_{11}q) + \tau_1 k\right)$$

$$H_{km}^R\left(y_1\left(1 - \frac{qb_1}{y_1} + \frac{c_1R_{22}}{(R_{11}R_{22} - R_{12}{}^2)y_1q}\right), y_2\left(1 - \frac{c_1R_{12}}{(R_{11}R_{22} - R_{12}{}^2)y_2q}\right), qe^{\tau_1}, s\right)$$

$$= \sum_{l=0}^{\infty} A_{lk}(g)H_{lm}^{(R)}(y_1,y_2)q^{l-k}, k,l = 0,1,2,\ldots . (2.14)$$

The matrix element $A_{lk}(g)$ are given by ([7]; p.87(4.26))

$$A_{lk}(g) = \exp(a_1 + k\tau_1)c_1^{k-l}L_l^{k-l}(-b_1c_1), k,l \geq 0. (2.15)$$

Substituting (2.15) into (2.14) and simplifying, we obtain the generating relation

$$\exp\left(\frac{qb_1}{2}(2R_{11}y_1 + 2R_{12}y_2 - b_1R_{11}q)\right)$$

$$H_{km}^R\left(y_1\left(1 - \frac{qb_1}{y_1} + \frac{c_1R_{22}}{(R_{11}R_{22} - R_{12}{}^2)y_1q}\right), y_2\left(1 - \frac{c_1R_{12}}{(R_{11}R_{22} - R_{12}{}^2)y_2q}\right)\right)$$

$$= \sum_{l=0}^{\infty} c_1^{k-l}L_l^{k-l}(-b_1c_1)H_{lm}^{(R)}(y_1,y_2)q^{l-k}, k = 0,1,2,\ldots . (2.16)$$

Again taking the operators (2.9) and proceeding exactly as before, we obtain the generating relation

$$\exp\left(\frac{sb_2}{2}(2R_{12}y_1 + 2R_{22}y_2 - b_2R_{22}s)\right)$$

$$H_{nr}^R\left(y_1\left(1 - \frac{c_2R_{12}}{(R_{11}R_{22}-R_{12}{}^2)y_1s}\right), y_2\left(1 - \frac{sb_2}{y_2} + \frac{c_2R_{11}}{(R_{11}R_{22}-R_{12}{}^2)y_2s}\right)\right)$$

$$= \sum_{l=0}^{\infty} c_2^{r-i}L_i^{r-i}(-b_2c_2)H_{ni}^{(R)}(y_1,y_2)s^{i-r}, r = 0,1,2,\ldots . (2.17)$$

3. Special Cases

We consider some special cases of generating relations (2.16) and (2.17).

1. In the quantum mechanical application if we consider the case of the absence of an extranal force i.e, when $y = 0$ and using eq.(1.5) in (2.16), we obtain the generating relation

$$\exp(-b_1R_{11}q))H_{km}^R(0,0) = \sum_{l=0}^{\infty} c_1^{k-l}L_l^{k-l}(-b_1c_1)H_{lm}^{(R)}(0,0)q^{l-k}, k = 0,1,2,\cdots . (3.1)$$

Similar results can be obtain for (2.17).

1. Using relation (1.6) in generating relation (3.1), we get the relation between Associated Lagender functions and Lagurre polynomials,

$$\exp(-b_1R_{11}q))\mu_{mk}!(-1)^{\frac{m+k}{2}}R_{11}^{\frac{k}{2}}R_{22}^{\frac{m}{2}}(r^2 - 1)^{\frac{m+k}{4}}P_{(m+k)/2}^{(m-k)/2}\left(\frac{r}{\sqrt{r^2-1}}\right).$$

$$= \sum_{l=0}^{\infty} c_1^{k-l}L_l^{k-l}(-b_1c_1)H_{lm}^{(R)}(0,0)q^{l-k}, k = 0,1,2,\cdots . (3.2)$$

Further for coinciding indices using (1.7) in relation (3.2) we obtain the relation between Lagender and Lagurre polynomials,

$$\exp(-b_1R_{11}q))n!(-detR)^{\frac{n}{2}}P_n\left(\frac{-R_{12}}{\sqrt{-detR}}\right)$$

$$= \sum_{n=0}^{\infty} c_1^{k-n}L_l^{k-n}(-b_1c_1)H_{nn}^{(R)}(0,0)q^{n-k}, k = 0,1,2,\cdots . (3.3)$$

where $P_n(z)$ being the Legender polynomial.

Similar results can be obtain for (2.17).

1. Using relation(1.10) in generating relation (2.16) we obtain the expression for $H_{nm}^{(R)}(y_1,y_2)$ in terms of $H_{nm}^{(R)}(0,0)$ and variables f_1, f_2 defined by Eq.(1.9)

$$\exp\left(\frac{qb_1}{2}(2R_{11}y_1 + 2R_{12}y_2 - b_1R_{11}q)\right)$$

$$H_{km}^R\left(y_1\left(1 - \frac{qb_1}{y_1} + \frac{c_1R_{22}}{(R_{11}R_{22} - R_{12}{}^2)y_1q}\right), y_2\left(1 - \frac{c_1R_{12}}{(R_{11}R_{22} - R_{12}{}^2)y_2q}\right)\right)$$

$$= \sum_{l=0}^{\infty} c_1{}^{k-l} L_l^{k-l}(-b_1 c_1) \sum_{u=0}^{l} \sum_{v=0}^{m} {}_l{}_u {}_m{}_v H_{uv}^{(R)}(0,0) f_1^{l-u} f_2^{m-v}, k = 0,1,2,\cdots.(3.4)$$

where

$$f_1 = R_{11}\left(y_1\left(1 - \frac{qb_1}{y_1} + \frac{c_1 R_{22}}{(R_{11}R_{22} - R_{12}{}^2)y_1 q}\right) + R_{12}y_2\left(1 - \frac{c_1 R_{12}}{(R_{11}R_{22} - R_{12}{}^2)y_2 q}\right)\right)$$

$$f_2 = R_{12}\left(y_1\left(1 - \frac{qb_1}{y_1} + \frac{c_1 R_{22}}{(R_{11}R_{22} - R_{12}{}^2)y_1 q}\right) + R_{22}y_2\left(1 - \frac{c_1 R_{12}}{(R_{11}R_{22} - R_{12}{}^2)y_2 q}\right)\right)$$

1. Further we can find relations for the case when $R_{22} = 0$ and $R_{11} = 0$ from relation (2.16) respectively as follows

$$\exp\left(\frac{qb_1}{2}(2R_{11}y_1 + 2R_{12}y_2 - b_1 R_{11}q)\right) H_{km}^R\left(y_1\left(1 - \frac{qb_1}{y_1}\right), y_2\left(1 + \frac{c_1 R_{12}}{R_{12}{}^2 y_2 q}\right)\right)$$

$$= \sum_{l=0}^{\infty} c_1{}^{k-l} L_l^{k-l}(-b_1 c_1) H_{lm}^{(R)}(y_1, y_2) q^{l-k}, k = 0,1,2,\cdots.(3.5)$$

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and

$$\exp(2R_{12}y_2)H_{km}^R\left(y_1\left(1 - \frac{qb_1}{y_1} - \frac{c_1 R_{22}}{R_{12}{}^2 y_1 q}\right), y_2\left(1 + \frac{c_1 R_{12}}{R_{12}{}^2 y_2 q}\right)\right)$$

$$= \sum_{l=0}^{\infty} c_1{}^{k-l} L_l^{k-l}(-b_1 c_1) H_{lm}^{(R)}(y_1, y_2) q^{l-k}, k = 0,1,2,\cdots.(3.6)$$

Similar results can be obtain for (2.17).

1. Taking $R_{11} = R_{22} = 0$,$R_{12} = 0$,$y_2 = s = 0$, $y_1 = x$ and replacing R_{12} and R_{22} by $-R_{12}$ and $-R_{22}$ respectively in relation (2.16), we obtain

$$\exp(-2b_1 x q - b_1{}^2 q^2))H_k^R\left(x\left(1 + \frac{qb_1}{x} - \frac{c_1}{2xq}\right)\right)$$

$$= \sum_{l=0}^{\infty} c_1{}^{k-l} L_l^{k-l}(-b_1 c_1) H_l^{(R)}(x) q^{l-k}, k = 0,1,2,\cdots.(3.7)$$

which is a result of Miller ([7]);p.106(4.76)).

Similar results can be obtain for (2.17).

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